

An inverse problem for the heat equation in view of practical application

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In this talk, we discuss an inverse problem to detect an inclusion in a homogeneous medium. As an approach to this problem, the application of the X-ray tomography is being studied, where the reconstruction is performed to reconstruct the object, by its section by section. With application of the X-ray tomography, the following problems are under investigation; non-destructive testing for mixed materials of the two kinds, non-destructive testing for the fuel tank of the rockets, non-destructive testing for die casting of the aluminium and so on. The problem of non-destructive testing for mixed materials of the two kinds arose from the development of the three dimensional CAD system which would enable us to describe the inner structure of the pillars and the walls in the buildings. In this problem, it is necessary to investigate the internal structure of the pillars consisted of the steel and the aluminium, which is not clear from their production process. It seems that it is not difficult to understand the motivation to study the other two examples, which are typical problems in non-destructive testing. For the time being, the same algorithm as the computerized tomography (CT) is applied to all of the above examples. Since the objects in these problems are much simpler than the interior structure of the human body, it is expected to reduce the X-ray data for the reconstruction of the object. This problem is closely related to the geometric tomography and there are many studies on it both in the viewpoint of theory and in the viewpoint of application. For example, confer [1, 6, 7] for the results in the viewpoint of theory and [2, 8, 9, 10] for the studies in the viewpoint of practical application. Unfortunately, the results mentioned above are not still satisfactory for practical application in view of the following points.

- In the case where we project parallel beams of the X-ray from two directions, we can classify the shape of the inclusions into the two classes, one is the uniquely determined ones by these data and the other is non-uniquely determined one ([6, 7, 8]). For the unique class, reconstruction formulas ([6, 8]) are given and we gave further studies, treatment of the errors, construction of a reconstruction algorithm and its implementation by computers and so on, satisfactory for practical application ([2, 8]). It is, however, proved that there are very few sets reconstructed by this method ([9]) and it is not known how to find the exact two directions for the reconstruction for the uniquely reconstructed sets, even if they exist.
- In addition to the above remark, since they apply cone beams of the X-ray in most industrial CT devices, we have to develop the counterpart of the above theory for the cone beams.

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- For general inclusions, the exact data of the beams of the X-ray for the reconstruction are not known. Needless to say their reconstruction methods, treatment of the errors, construction of an approximate reconstruction algorithm, its implementation by computers and so on.

There are other problems of the use of the X-ray tomography.

- (i) The cost of the testing is very expensive if we apply the X-ray tomography.
- (ii) We cannot ignore harmful influence of the X-ray on the human body.

In order to solve the problems (i) and (ii), we try another approach. We study to detect an inclusion in a homogeneous medium applying the heat conduction. It was M.Ikehata and M.Kawashita [3, 4, 5] who developed the study began to study to detect an inclusion in a homogeneous medium applying the heat conduction. They studied the following problem.

Problem 1. *Let Ω be a bounded domain of \mathbb{R}^n , $n = 2, 3$, with smooth boundary. Let D be an open subset of Ω with smooth boundary and satisfy that $\overline{D} \subset \Omega$ and $\Omega \setminus D$ is connected. We denote the unit outward normal vectors to $\partial\Omega$ and ∂D by the same symbol ν . Let $T > 0$ be an arbitrary. Given $f = f(x, t)$, $(x, t) \in \partial\Omega \times (0, T)$, let $u = u(x, t)$ be the solution of the initial boundary value problem for the heat equation*

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } (\Omega \setminus D) \times (0, T), \\ \partial_\nu u = 0 & \text{on } \partial D \times (0, T), \\ \partial_\nu u = f & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = 0 & \text{in } \Omega \setminus D. \end{cases} \quad (1)$$

In this case, is it possible to reconstruct D by the boundary data $u|_{\partial\Omega}$ if we suitably control the heat flux f ?

This is an inverse problem to apply “Neumann to Dirichlet” boundary data. They proved that the convex hull, as well as some other information, of the inclusion D is reconstructed with the choice of a suitable adjoint solution of the heat equation. Their theory being very excellent and beautiful as mathematical one, it seems that there are several points to be modified in view of practical application.

- In practical application, it is not easy give the heat flux as the boundary data. In addition to it, its observation is not easy, either.
- Though Ikehata-Kawashita controlled the input of the heat (flux) $f(x, t)$ on the whole boundary points $x \in \partial\Omega$, in view of the practical application, it is much easier to give only one point source $\delta(x_0)f(t)$ on a fixed boundary point $x_0 \in \partial\Omega$.

In view of these remarks, we study the following problem.

Problem 2. *Let Ω be a bounded domain of \mathbb{R}^n $n = 2, 3$ with smooth boundary. Let D be an open subset of Ω with smooth boundary and satisfy that $\overline{D} \subset \Omega$ and $\Omega \setminus D$ is connected. We denote the unit outward normal vectors to $\partial\Omega$ and ∂D by the same symbol ν . Let*

$T > 0$ be an arbitrary. Given $f = f(t)$, $t \in \times(0, T)$ and $x_0 \in \partial\Omega$, let $u = u(x, t)$ be the solution of the initial boundary value problem for the heat equation

$$\begin{cases} \partial_t u - \Delta u = \delta(x_0)f(t) & \text{in } (\Omega \setminus D) \times (0, T), \\ \partial_\nu u = 0 & \text{on } \partial D \times (0, T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = 0 & \text{in } \Omega \setminus D. \end{cases} \quad (2)$$

In this case, is it possible to reconstruct D by the boundary data $u|_{\partial\Omega}$ if we suitably control the point heat source $f(t)$ at $x_0 \in \partial\Omega$?

It is our main purpose in this talk to study Problem 2. For this purpose, we apply the idea of hyperfunctions to treat the Delta functions on the boundary $\partial\Omega$. Even if the reconstruction formulas for the inclusions are obtained, the known results on Problems 1 and 2 are far from being applied for practice. At the end of this talk, we mention open problems to be solved in order that the studies of these problems should be applied for practice.

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