Process-independent effective charge: from QCD's Green's functions to Hadron phenomenology

Daniele Binosi, Cedric Mezrag, Joannis Papavassiliou, Craig D. Roberts, José Rodríguez-Quintero, K. Raya, J. Segovia, F. de Soto ...

[Few Body Syst. 59 (2018) no.6, 121]
Preliminary results...

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QCD's running coupling. Motivation.

The QCD Holy Grail: the understanding of hadrons in terms of its elementary excitations; namely, quarks and gluons!

What happens down here?
QCD's running coupling. Motivation.

The QCD Holy Grail: the understanding of hadrons in terms of its elementary excitations; namely, quarks and gluons!

**Confinement**

Colored bound states have never been seen to exist as particles in nature.

**DCSB**

Chiral symmetry appears dynamically violated in the Hadron spectrum.

Emergent phenomena playing a dominant role in the real world dominated by the IR dynamics of QCD.
Quark's gap equation

Let's start by the beginning:

- Dynamical chiral symmetry breaking generates the "constituent-quark" masses
- This is the most important mechanism for the mass generation in our Universe (responsible for around 98% of the proton mass)
- The effect is realized through the quark's gap equation and is clearly achieved through the pure theory's dynamics (nothing needs to be added to QCD!!)
Quark's gap equation

Beyond Rainbow-Ladder truncation:
One-gluon exchange effective kernel + Tree-level quark-gluon vertex

\[
S^{-1}(p) = Z_2 (iγ \cdot p + m^{bm}) + \Sigma(p),
\]

\[
\Sigma(p) = Z_1 \int_{dq} g^2 D_{\mu\nu}(p-q) \frac{\lambda^a}{2} γ_\mu S(q) \frac{\lambda^a}{2} (Γ_\nu(q, p)),
\]

\[
4π I(k^2) \frac{1}{k^2} \left( δ_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)
\]

\[
I(k^2) = k^2 \frac{G_{IR}(k^2) + G_{UV}(k^2)}{4π}
\]

\[
G_{IR}(k^2) = \frac{8π^2}{ω^5} ζ^3 e^{-k^2/ω^2}
\]

\[
G_{UV}(k^2) = \frac{96π^2}{25} \frac{1 - e^{-k^2/[GeV^2]}}{k^2 log[e^2 - 1 + (1 + k^2/Λ^2)^2]}
\]

\[
Γ_\mu = Γ_{BC}^{ACM} + Γ_{ACM}^{ACM}
\]

- Ball-Chiu vertex [PRD(22)1980]
- Anomalous Chromomagnetic vertex
Consistent with both linear and transverse STI

Model parameters:
\[
Λ = 0.234 \text{ GeV}
\]
\[
ζ = 0.55 \text{ GeV}
\]
\[
ω ∈ [0.4, 0.6] \text{ GeV}
\]

Fixed by the pion decay constant
Quark's gap equation

Beyond Rainbow-Ladder truncation:
One-gluon exchange effective kernel + Tree-level quark-gluon vertex
Quark's gap equation: RGI interaction

- **Universal (process-independent) contribution:**
  originates entirely from the gauge sector

**Fundamental quantities:** PT-BFM propagators/vertices
satisfy Abelian-like Slavnov-Taylor (ST) identities

**How to get them?**
use the PT algorithm
Cornwall, Papavassiliou, PRD 40 (1989)

\[
\hat{\Gamma}^{\alpha\mu\nu} = (k_2 - k_1)^{\alpha\mu} + 2q^{\nu} g^{\alpha\mu} - 2q^{\mu} g^{\alpha\nu}
\]
\[
\Gamma_P^{\alpha\mu\nu} = k_1^\mu g^{\alpha\nu} - k_2^\nu g^{\alpha\mu}
\]
  • longitudinal momenta
  trigger elementary Ward identities

**Apply the PT to the quark-gluon vertex**
one loop result:
Quark's gap equation: RGI interaction

- **Allot pieces to different Green's functions**
  - Construct $\hat{\Delta}$ and $\hat{\Gamma}_\mu$

- **Crucial all-order equivalence: PT=BFM**
  - Yields Feynman rules for systematic calculation
  - $\hat{\Delta} \sim \frac{1}{q^2[1 + b g^2 \log q^2/\mu^2]}$; $b = 11C_A/48\pi^2$
  - Absorbs all the RG logs as the photon in QED
  - Renormalizes as $Z_g^{-2}$

- **An additional equivalence holds: antiBRST+BRST=BFM**
  - Plethora of symmetry identities, in particular BQ identities
  - $\Delta(q^2) = [1 + G(q^2)]^2 \hat{\Delta}(q^2)$
  - $G$ special PT-BFM function:
    - Determined by ghost-gluon dynamics
  - Combination $1+G$ appears in all BQIs
    - Fundamental non-Abelian quantity
  - $G$ is related (Landau gauge) to the ghost dressing:
    - Use ghost gap equation to constrain $1+G, L$
    - $F^{-1}(q^2) = 1 + G(q^2) + L(q^2)$
Quark's gap equation: RGI interaction

Convert vertices/propagators into PT-BFM ones
new RG invariant combination appears
\[ \tilde{d}(k^2) = \alpha(\mu^2)\Delta(k^2; \mu^2) \]

Use symmetry identity
to identify the interaction strength
Aguilar, DB, Papavassiliou, Rodriguez-Quintero, PRD 90 (2009)
DB, Chang, Papavassiliou, Roberts, PLB 742 (2015)
\[ I(k^2) = k^2 \tilde{d}(k^2) \]
\[ \tilde{d}(k^2) = \frac{\alpha(\mu^2)\Delta(k^2; \mu^2)}{[1 + G(k^2; \mu^2)]^2} \]

1+G and \( L \) determined by their own SDEs
under simplifying assumptions:
\[ 1 + G(p^2) = Z_c - g^2 \int k \left[ 2 + \frac{(k \cdot p)^2}{k^2 p^2} \right] \frac{B_1(k)\Delta(k)F((k + p)^2)}{(k + p)^2}, \]
\[ L(p^2) = -g^2 \int k \left[ 1 - 4 \frac{(k \cdot p)^2}{k^2 p^2} \right] \frac{B_1(k)\Delta(k)F((k + p)^2)}{(k + p)^2}. \]
\[ F^{-1}(q^2) = Z_c - 3g^2 \int k \left[ 1 - \frac{(k \cdot p)^2}{k^2 p^2} \right] \frac{B_1(k)\Delta(k)F((k + p)^2)}{(k + p)^2} \]

- Main source of uncertainties:
needs assumptions on ghost vertex behavior
- Parametrized by \( \delta \in [0, 1] \)
lower bound \( \delta = 0 \): \( 1/F = 1+G \)
Quark's gap equation: RGI interaction

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1+G and L determined by their own SDEs
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\[
1 + G(p^2) = Z_c - g^2 \int_k \left[ 2 + \frac{(k \cdot p)^2}{k^2 p^2} \right] B_1(k) \Delta(k) \frac{F((k + p)^2)}{(k + p)^2}
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\]
\[
F^{-1}(q^2) = Z_c - 3 \, g^2 \int_k \left[ 1 - \frac{(k \cdot p)^2}{k^2 p^2} \right] B_1(k) \Delta(k) \frac{F((k + p)^2)}{(k + p)^2}
\]

Both top-bottom and down-up approaches deliver quark-gluon effective interactions which compare remarkably well with each other!
QCD effective charge

Let us now carefully examine the RGI Interaction:

\[ I(k^2) := k^2 \hat{d}(k^2) = \frac{\alpha_T(k^2)}{[1 - \frac{\alpha_T(k^2)}{L(k^2)F(k^2)}]^2} \]

\[ L(k^2; \zeta^2) \approx \frac{3g^2(\zeta^2)}{32\pi^2} \left( \frac{\ln \frac{k^2}{\Lambda_T^2}}{\ln \frac{\zeta^2}{\Lambda_T^2}} \right)^{-(\gamma_0 + \gamma_0)/\beta_0} \]

\[ F(k^2; \zeta^2) \approx \left( \ln \frac{k^2}{\Lambda_T^2} / \ln \frac{\zeta^2}{\Lambda_T^2} \right)^{-\gamma_0/\beta_0} \]

\[ L(k^2)F(k^2) \approx \frac{3}{2\beta_0 \ln (k^2/\Lambda_T^2)} \]

\[ \alpha_{MS}(k^2)(1 + 1.09 \alpha_{MS}(k^2) + ...) \]

Let us now carefully examine the RGI Interaction:

\[
I(k^2) := k^2 \hat{d}(k^2) = \frac{\alpha_T(k^2)}{1 - \left( \frac{\alpha_T(k^2)}{L(k^2) F(k^2)} \right)^2}
\]

\[
L(k^2; \zeta^2) \approx \begin{cases} 
3g^2(\zeta^2) & \frac{k^2}{\Lambda_T^2} \gg 1 \\
\left( \frac{\ln \frac{k^2}{\Lambda_T^2} / \ln \frac{\zeta^2}{\Lambda_T^2}}{32\pi^2} \right)^{-(\tilde{\gamma}_0 + \gamma_0) / \beta_0} & \frac{k^2}{\Lambda_T^2} \ll 1 
\end{cases}
\]

\[
I(k^2) \approx k^2 \hat{d}(0) \left[ 1 - \left( \frac{\hat{d}(0)}{8\pi} + \frac{\ell_w}{m_g^2} \right) k^2 \ln \frac{k^2}{\Lambda_T^2} \right]
\]

\[
L(k^2) F(k^2) \approx \frac{3}{2\beta_0 \ln (k^2 / \Lambda_T^2)}
\]

\[
\hat{d}(0) = \frac{\alpha_0}{m_0^2} \approx \frac{0.9\pi}{(m_P/2)^2}
\]

\[
\alpha_{MS}(k^2)(1 + 1.09 \alpha_{MS}(k^2) + \ldots)
\]
QCD effective charge

Let us first carefully examine the RGI Interaction:

\[ I(k^2) := k^2 \hat{d}(k^2) = \frac{\alpha_T(k^2)}{[1 - L(k^2)F(k^2)]^2} \]

Remarkable QCD feature: saturation of the RG key ingredient \( \hat{d}(0) \)

Define then the RGI invariant function

\[ \hat{d}(0) = \frac{\alpha_0}{m_0^2} \approx \frac{0.9\pi}{(m_P/2)^2} \]

Extract the (process-independent) coupling

Using the quark gap equation
QCD effective charge: comparison.

- Process dependent effective charges fixed by the leading-order term in the expansion of a given observable

- Bjorken sum rule defines such a charge
  Bjorken, PR 148 (1966); PRD 1 (1970)
  \[
  \int_0^1 dx \left[ g_1^p(x, k^2) - g_1^n(x, k^2) \right] = \frac{g_A}{6} \left[ 1 - \alpha_g(k^2)/\pi \right]
  \]
  - $g_1^{p,n}$ spin dependent p/n structure functions extracted from measurements using unpolarized targets
  - $g_A$ nucleon flavour-singlet axial charge

- Many merits
  - Existence of data for a wide momentum range
  - Tight sum rules constraints on the integral at IR and UV extremes
  - Isospin non-singlet suppress contributions from hard-to-compute processes

- Equivalence in the perturbative domain reasonable definitions of the charge
  \[
  \alpha_{g_1}(k^2) = \alpha_{MS}(k^2)[1 + 1.14\alpha_{MS}(k^2)] + \cdots
  \]
  \[
  \bar{\alpha}_{PT}(k^2) = \alpha_{MS}(k^2)[1 + 1.09\alpha_{MS}(k^2)] + \cdots
  \]

- Agreement with light-front holography model for $\alpha_{g_1}$
  Deur, Brodsky, de Teramond, PPNP 90 (2016)
Pl-effective charge from lattice data with Nf=3 flavors at the physical point

Preliminary results:

\[
\hat{\alpha}_{\text{PI}}(q^2) = \frac{\hat{d}(q^2)}{\mathcal{D}(q^2)} \approx \frac{\alpha_T(q^2)}{q^2 [1 - L(q^2, \zeta^2) F(q^2, \zeta^2)]^2} \frac{m_0^2 \Delta_F(0, \zeta^2)}{\Delta_F(q^2, \zeta^2)}
\]

\[
= \alpha_T(\zeta^2) \frac{F(q^2, \zeta^2)}{[1 - L(q^2, \zeta^2) F(q^2, \zeta^2)]^2} \Delta_F(0, \zeta^2) m_0^2
\]

The IR running of the PI effective charge with momenta only depends on:
- The ghost dressing function
PI-effective charge from lattice data with Nf=3 flavors at the physical point

Preliminary results:

$$\hat{\alpha}_{\text{PI}}(q^2) = \frac{\hat{d}(q^2)}{D(q^2)} \approx \frac{\alpha_T(q^2)}{q^2 \left[1 - L(q^2, \zeta^2)F(q^2, \zeta^2)\right]^2} \frac{m_0^2 \Delta_F(0, \zeta^2)}{\Delta_F(q^2, \zeta^2)}$$

$$= \alpha_T(\zeta^2) \frac{F(q^2, \zeta^2)}{[1 - L(q^2, \zeta^2)F(q^2, \zeta^2)]^2} \Delta_F(0, \zeta^2) m_0^2$$

The IR running of the PI effective charge with momenta only depends on:

- The ghost dressing function
- The PT-BFM function L
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= \alpha_T(\zeta^2) \frac{F(q^2, \zeta^2)}{[1 - L(q^2, \zeta^2)F(q^2, \zeta^2)]^2} \Delta_F(0, \zeta^2)m_0^2
\]

The IR running of the PI effective charge with momenta only depends on:

- The ghost dressing function
- The PT-BFM function \( L \)

Its strength depends also on the saturation point at zero-momentum of the gluon propagator.
Pl-effective charge from lattice data with Nf=3 flavors at the physical point

Preliminary results:

\[ \hat{\alpha}_{\text{PI}}(q^2) = \frac{d(q^2)}{D(q^2)} \approx \frac{\alpha_T(q^2)}{q^2 \left[ 1 - L(q^2, \zeta^2) F(q^2, \zeta^2) \right]^2} \frac{m_0^2 \Delta_F(0, \zeta^2)}{\Delta_F(q^2, \zeta^2)} \]

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The IR running of the PI effective charge with momenta only depends on:

- The ghost dressing function
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Its strength depends also on the saturation point at zero-momentum of the gluon propagator and on the Taylor coupling.
PI-effective charge from lattice data with Nf=3 flavors at the physical point

\[ \hat{\alpha}_{\text{PI}}(q^2) = \frac{\hat{d}(q^2)}{D(q^2)} \approx \frac{\alpha_T(q^2)}{q^2 [1 - L(q^2, \zeta^2) F(q^2, \zeta^2)]^2} \frac{m_0^2 \Delta_F(0, \zeta^2)}{\Delta_F(q^2, \zeta^2)} \]

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Preliminary results:

All put together:

Less uncertainties (that of the gluon mass is only left here) and still a better agreement with the world data for the experimental determination of the Bjorken sum-rule effective charge.

The IR running of the PI effective charge with momenta only depends on:
- The ghost dressing function
- The PT-BFM function L

Its strength depends also on the saturation point at zero-momentum of the gluon propagator and on the Taylor coupling.
One application: pion PDF DGLAP evolution

The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator

\[ q^\pi(x; \zeta) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \left| \overline{\psi}(-z)\gamma^+\psi(z) \right| P \right\rangle \bigg|_{z^+=0, z_\perp=0} \]
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The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

\[ q^\pi (x, \xi_H) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P|\bar{\psi}^u(-z)\gamma^+\psi^q(z)|P\right\rangle \bigg|_{z^+=0, z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_u(x, k_\perp) \Psi^*_u(x, k_\perp) \]
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\]

LFWF leading to asymptotic PDAs

\[
q_{sf}(x) \approx 30 x^2 (1 - x)^2
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A more realistic pion 2-body LFWF

DCSB-induced hardening
One application: pion PDF DGLAP evolution

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\[ q^\pi(x; \zeta_H) = \frac{1}{2} \int \frac{dz^-.}{2\pi} e^{ixP+z} \left\langle P \left| \bar{\psi}^q(-z) \gamma^+ \psi^q(z) \right| P \right\rangle \bigg|_{z^+ = 0, z_\perp = 0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi_{uJ}^*(x, k_\perp) \Psi_{uJ}(x, k_\perp) \]

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A more realistic pion 2-body LFWF

Direct computation of Mellin moments:

\[ (x^m)^\pi_{\zeta_H} = \int_0^1 dx \, x^m \, q^\pi(x; \zeta_H) \]

\[ = \frac{N_c}{n \cdot P} \text{tr} \int dk \left[ \frac{n \cdot k_\eta}{n \cdot P} \right]^m \Gamma_\pi(k_\eta, P) S(k_\eta_n) n \cdot \partial_{k_\eta} \left[ \Gamma_\pi(k_\eta, -P) S(k_\eta) \right] \]
An application: pion PDF DGLAP evolution

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\[ q^\pi(x; \zeta_H) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{i x P^+ z^-} \left\langle P \left| \bar{\psi}^q(-z) \gamma^+ \psi^q(z) \right| P \right\rangle \bigg|_{z^+ = 0, z_- = 0} = \int \frac{d^3 k_\perp}{16 \pi^3} \frac{1}{P} \Psi^\ast_{u J}(x, k_\perp) \Psi_{u J}(x, k_\perp) \]

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\[ x \cdot q^\pi(x; \zeta_H) = 213.32 x^2 (1 - x)^2 \times \left[ 1 - 2.9342 \sqrt{x(1 - x)} + 2.2911 x(1 - x) \right] \]
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\[
q^\pi(x; \zeta_H) = \frac{1}{2} \int \frac{dz^+}{2\pi} e^{ix \cdot P + z^-} \left\langle P \left| \bar{\psi}^u(-z) \gamma^+ \psi^q(z) \right| P \right\rangle \bigg|_{z^+, z^- = 0} = \int \frac{d^2 k_\perp}{16\pi^3} \Psi_{uJ}^*(x, k_\perp) \Psi_{uJ}(x, k_\perp)
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LFWF leading to asymptotic PDAs

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LFWF leading to asymptotic PDAs
\[ q_{sf}(x) \approx 30 x^2 (1 - x)^2 \]

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\[ (x^m)^\pi_{\zeta_H} = \int_0^1 dx x^m q^\pi(x; \zeta_H) = \frac{N_c}{n \cdot P} \text{tr} \int dk \left[ n \cdot k_\eta \right]^m \Gamma_\pi(k_\eta, P) S(k_\eta) n \cdot \partial_{k_\eta} [\Gamma_\pi(k_\eta, -P) S(k_\eta)] \]

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q^\pi(x; \zeta_H) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^{+}z^-} \left\langle P \left| \bar{\psi}^q(-z)\gamma^+\psi^q(z) \right| P \right\rangle \bigg|_{z^+=0, z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_u(x, k_\perp) \Psi_u(x, k_\perp) \quad \text{LFWF leading to asymptotic PDAs}
\]

\[
\zeta_H \rightarrow \zeta_2 = 5.2 \text{ GeV}
\]

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= \frac{N_c}{n \cdot P} \, \text{tr} \int_d k \left[ \frac{n \cdot k_\eta}{n \cdot P} \right]^m \Gamma_\pi(k_\eta, P) S(k_\eta) n \cdot \partial_{k_\eta} [\Gamma_\pi(k_\eta, -P) S(k_\eta)]
\]

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q^\pi(x; \zeta_H) = 213.32 \, x^2 (1-x)^2 \times [1 - 2.9342 \sqrt{x(1-x)} + 2.2911 x(1-x)]
\]
One application: pion PDF DGLAP evolution

\[ M_n(t) = \int_0^1 dx \ x^n q(x, t) \]
\[ t = \ln \left( \frac{\zeta}{\zeta_0}^2 \right) \]

Moments' evolution (1-loop):

\[ \frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4 \pi} \gamma_0^n M_n(t) + \ldots \]
One application: pion PDF DGLAP evolution

A master equation for the (1-loop) moments' evolution:

\[
\frac{d}{dt} q(x,t) = -\frac{\alpha(t)}{4\pi} \int_0^{1} \frac{dy}{y} q(y,t) P\left(\frac{x}{y}\right) + \ldots
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\[ \frac{d}{dt} M_n(t) = - \frac{\alpha(t)}{4 \pi} \gamma_0^n M_n(t) + \ldots \]

\[ P(x) = \frac{8}{3} \left( \frac{1 + z^2}{(1-x)_+} + \frac{3}{2} \delta(x-1) \right) \]

\[ \gamma_n = - \frac{4}{3} \left( 3 + \frac{2}{(n+2)(n+3)} \right) - 4 \sum_{i=1}^{n+1} \frac{1}{i} \]
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\[
\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots
\]

\[
t_\Lambda = \ln\left(\frac{\Lambda^2}{\zeta_2}\right)
\]

**One application:** pion PDF DGLAP evolution

**Moments' evolution (1-loop):**

\[
M_n(t) = \int_0^1 dx x^n q(x, t)
\]

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t = \ln\left(\frac{\zeta_2^2}{\zeta_0^2}\right)
\]

\[
P(x) = \frac{8}{3} \left(\frac{1+z^2}{(1-x)_+} + \frac{3}{2} \delta(x-1)\right)
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\]

\[
\zeta_{\text{fit}} = 5.2 \text{ GeV}
\]
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\[
\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots
\]

\[
t_\Lambda = \ln\left(\frac{\Lambda^2}{\zeta_0^2}\right)
\]

\[
M_n(t) = M_n(t_0) \left(\frac{\alpha(t)}{\alpha(t_0)}\right)^{\gamma_0^n/\beta_0}
\]

One application: pion PDF DGLAP evolution

\[
\int_0^1 dx \cdot P(x) = \gamma_0^n
\]

\[
P(x) = \frac{8}{3} \left(\frac{1+z^2}{(1-x)_+} + \frac{3}{2} \delta(x-1)\right)
\]

\[
\gamma_0^n = -\frac{4}{3} \left(3 + \frac{2}{(n+2)(n+3)} - 4 \sum_{i=1}^{n+1} \frac{1}{i}\right)
\]

\[
\zeta_H = \zeta_2 = 5.2 \ \text{GeV}
\]

\[
M_n(t) = \int_0^1 dx \ x^n q(x, t)
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\[
t = \ln\left(\frac{\zeta_2^2}{\zeta_0^2}\right)
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One application: pion PDF DGLAP evolution

Which value of Lambda?

\[ \alpha(t) = \frac{4\pi}{\beta_0(t-t\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln \left( \frac{\zeta^2}{\Lambda^2} \right)} + \ldots \]
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Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

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\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\xi^2}{\Lambda^2}\right)} + \ldots
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\ln\left(\frac{\Lambda^2}{\bar{\Lambda}^2}\right) = \frac{4\pi}{\beta_0} \left(\frac{1}{\alpha(t)} - \frac{1}{\bar{\alpha}(t)}\right) + \ldots = \frac{4\pi c}{\beta_0}
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\[ \alpha(t) = \bar{\alpha}(t)(1 + c\bar{\alpha}(t) + \ldots) \]

The evolution will thus depend on the scheme via the perturbative truncation.

\[ \frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma^n_0 M_n(t) + \ldots \]

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The evolution will thus depend on the scheme via the perturbative truncation and the usual prejudice is that truncation errors are optimally small in MS scheme.

\[ \frac{d}{dt} \frac{\alpha(t)}{4\pi} = -\frac{\alpha^2(t)}{4\pi} \beta_0 + \ldots \]

PDG2018: [PRD98(2018)030001]

\[ \begin{align*}
\Lambda_{MS}^{(5)} &= (210 \pm 14) \text{ MeV,} \\
\Lambda_{MS}^{(4)} &= (292 \pm 16) \text{ MeV,} \\
\Lambda_{MS}^{(3)} &= (332 \pm 17) \text{ MeV,}
\end{align*} \]
One application: pion PDF DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

\[
\langle x^m \rangle_{\zeta_H} = \int_0^1 dx \, x^m q^\pi(x; \zeta_H)
= \frac{N_c}{n \cdot P} \operatorname{tr} \int_{dk} \left[ \frac{n \cdot k}{n \cdot P} \right]^m \Gamma_\pi(k_\eta, P) S(k_\eta) n \cdot \delta_{k_\eta} \left[ \Gamma_\pi(k_\eta, -P) S(k_\eta) \right] 
\times [1 - 2.9342 \sqrt{x(1-x)} + 2.2911 x(1-x)]
\]

Optimal best-fitting parameters:
\[\Lambda_{QCD} = 0.234 \text{ GeV} ; \quad \zeta_H = 0.349 \text{GeV} .\]
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\]

\[q^\pi(x; \zeta_H) = 213.32 \, x^2 (1 - x)^2 \times [1 - 2.9342 \sqrt{x(1-x)} + 2.2911 \, x(1-x)]\]

\[\zeta_H \rightarrow \zeta_2 = 5.2 \text{ GeV}\]

Optimal best-fitting parameters:

\[\Lambda_{QCD} = 0.234 \text{ GeV} ; \]
\[\zeta_H = 0.349 \text{ GeV} .\]

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<tr>
<th>[\zeta_2]</th>
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Comparison with the three first moments obtained from lQCD
One application: pion PDF DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

\[ \langle x^m \rangle_{\zeta_H} = \int_0^1 dx \, x^m \, q^\pi (x; \zeta_H) \]
\[ = \frac{N_c}{n \cdot P} \text{tr} \left( \frac{n \cdot k_\eta}{n \cdot P} \right)^m \Gamma_\pi (k_\eta, P) S(k_\eta) n \cdot \delta_{k_\eta} [\Gamma_\pi (k_\eta, -P) S(k_\eta)] \]

\[ q^\pi (x; \zeta_H) = 213.32 x^2 (1 - x)^2 \times [1 - 2.9342 \sqrt{x(1 - x)} + 2.2911 x(1 - x)] \]

\[ \zeta_H \rightarrow \zeta_L = 2 \text{ GeV} \rightarrow \zeta_2 = 5.2 \text{ GeV} \]

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\[ \zeta_H = 0.349 \text{ GeV} . \]

\[ \Lambda_{QCD} = 0.234 \text{ GeV} ; \]
\[ \zeta_H = 0.374 \text{ GeV} . \]

Matching the three first moments obtained from lQCD

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One application: pion PDF DGLAP evolution

Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

$$\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\xi^2}{\Lambda^2}\right)} + \ldots$$

$$\ln\left(\frac{\Lambda^2}{\tilde{\Lambda}^2}\right) = \frac{4\pi}{\beta_0} \left( \frac{1}{\alpha(t)} - \frac{1}{\tilde{\alpha}(t)} \right) + \ldots = \frac{4\pi c}{\beta_0}$$

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The evolution will thus depend on the scheme via the perturbative truncation

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} y_0^n M_n(t) + \ldots$$

$$\frac{d}{dt} \alpha(t) = -\frac{\alpha^2(t)}{4\pi} \beta_0 + \ldots$$

The use of $\Lambda = 0.234\text{GeV}$ can be thus interpreted as the choice of particular scheme, differing from MS.
One application: **pion PDF DGLAP evolution**

Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

\[ \alpha(t) = \frac{4\pi}{\beta_0(t - t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0} \ln\left(\frac{\xi^2}{\Lambda^2}\right) + \ldots \]

\[ \ln\left(\frac{\Lambda^2}{\Lambda'^2}\right) = \frac{4\pi}{\beta_0} \left( \frac{1}{\alpha(t)} - \frac{1}{\bar{\alpha}(t)} \right) + \ldots = \frac{4\pi c}{\beta_0} \]

\[ \frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) \]

The evolution will thus depend on the scheme via the perturbative truncation.

The use of \( \Lambda = 0.234 \) GeV can be thus interpreted as the choice of particular scheme, differing from MS. Beyond this, the scheme can be defined in such a way that one-loop DGLAP is exact at all orders (Grunberg's effective charge).
One application: pion PDF DGLAP evolution

\[ \alpha(t) = \frac{4\pi}{\beta_0 \ln \left( \frac{m_\alpha^2 + \zeta_0^2 \exp(t)}{\Lambda^2} \right)} = \frac{4\pi}{\beta_0 \ln \left( \frac{m_\alpha^2 + k^2}{\Lambda^2} \right)} \]

\[ t = \ln \left( \frac{k^2}{\zeta_0^2} \right) \]
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\[ \alpha(0) = \alpha_{PL}(0) \rightarrow m_\alpha = 0.300 \text{ GeV} \]
One application: pion PDF DGLAP evolution

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\[ \alpha(0) = \alpha_{PL}(0) \rightarrow m_\alpha = 0.300 \text{ GeV} \]

\[ \frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) \]

Numerical integration with the effective charge

\[ M_n(t) = M_n(t_0) \exp \left( -\frac{\gamma_0^n}{4\pi} \int_{t_0}^{t} dz \alpha(z) \right) \]

\[ \gamma_0^n = -\frac{4}{3} \left( 3 + \frac{2}{(n+2)(n+3)} - \sum_{i=1}^{n+1} \frac{1}{i} \right) \]
\[ \alpha(t) = \frac{4\pi}{\beta_0 \ln \left( \frac{m_\alpha^2 + \zeta_0^2 \exp(t)}{\Lambda^2} \right)} = \frac{4\pi}{\beta_0 \ln \left( \frac{m_\alpha^2 + k^2}{\Lambda^2} \right)} \]

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If one identifies: \( m_\alpha \equiv \zeta_H \), all the scales (and the evolution between them) appear thus fixed, apart from \( \Lambda_{QCD} \) (fixed by the scheme).
Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

\[ \Lambda_{QCD} = 0.234 \text{ GeV}; \quad \zeta_H \equiv m_\alpha \rightarrow \zeta_2 = 5.2 \text{ GeV} \]

If one identifies: \( m_\alpha \equiv \zeta_H \), all the scales (and the evolution between them) appear thus fixed, apart from \( \Lambda_{QCD} \) (fixed by the scheme). And the agreement with E615 data is perfect!!!
One application: pion PDF DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

The same is obtained from the overlap of realistic pion 2-body LFWFs

and after integration of the DGLAP master equation

\[ \frac{d}{dt} q(x,t) = -\frac{\alpha(t)}{4\pi} \int_x^1 \frac{dy}{y} q(y,t) P(\frac{x}{y}) + \ldots \]

If one identifies: \( m_\alpha \equiv \zeta_H \), all the scales (and the evolution between them) appear thus fixed, apart from \( \Lambda_{QCD} \) (fixed by the scheme). And the agreement with E615 data is perfect!!!
Conclusions

- One can define a parameter-free effective charge (completely determined from 2-points gauge sector), with no Landau pole (physical coupling showing an IR fixed point) and smoothly connecting IR and UV domains (no explicit matching procedure).

- It is shown to be in good agreement with the Bjorken-rule effective charge and with the coupling from the light-front holographic model.

\[ \Lambda_{QCD} = 0.234 \text{ GeV}; \quad \zeta_H \equiv m_\alpha \to \zeta_2 = 5.2 \text{ GeV} \]

- The PI effective charge is shown to be equivalent, within the IR domain, to the one which can be defined to get an “exact” (at all order in perturbations) DGLAP evolution for the pion PDF.