Higher order Symmetric Cumulants

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Study the Properties of Quark-Gluon Plasma

➢ State of matter with deconfined quarks and gluons
➢ Present in
  ❖ our early Universe?
  ❖ the core of Neutron Stars?
Study the Properties of Quark-Gluon Plasma

- State of matter with deconfined quarks and gluons
- Present in
  - our early Universe?
  - the core of Neutron Stars?

Better understanding of the Universe by studying the properties of QCD matter at extreme conditions
Create Quark-Gluon Plasma on Earth

- Direct observation of QGP not possible
- Many probes to study QGP in heavy-ion collisions:
  - Anisotropic flow
  - Jet quenching
  - …

Study of heavy-ion collisions can provide access to the properties of the QGP
Anisotropies in the Initial and Final States

- Volume of interacting matter initially anisotropic in coordinate space
  \[
  \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \ldots
  \]

- Anisotropic flow: Transfer of the initial anisotropy into anisotropy in momentum space via the thermalized medium

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Anisotropies in the Initial and Final States

➢ Described by Fourier series:

\[ f(\varphi) = \frac{1}{2\pi} \left[ 1 + 2 \sum_{n=1}^{\infty} v_n \cos[n(\varphi - \Psi_n)] \right] \]


➢ \( v_n \) and \( \Psi_n \) related to \( \varphi \):

\[ \langle \cos[n_1\varphi_1 + \cdots + n_k\varphi_k] \rangle = v_{n_1} \cdots v_{n_k} \cos[n_1\Psi_{n_1} + \cdots + n_k\Psi_{n_k}] \]


\( \varphi \): Azimuthal angle
\( v_n \): Flow harmonic
\( \Psi_n \): Symmetry-plane

\[ \phi \]

\[ \Psi \]

\[ z \]

\[ y \]

\[ x \]

\[ \psi_3 \]
Flow and Properties of Quark-Gluon Plasma

- Measurements of $v_n$ sensitive to properties of QGP
- Correlations between two harmonics probed by:
  \[ \text{SC}(m, n) = \langle v_m^2 v_n^2 \rangle - \langle v_m \rangle^2 \langle v_n \rangle^2 \]
- New and independent set of constraints on the system

ALICE Collaboration, PRL 117, 182301 (2016)
Flow and Properties of Quark-Gluon Plasma

- Combinations of $v_n$ and $\Psi_n$ sensitive to properties of QGP
- Correlations between two harmonics probed by:
  \[ SC(m, n) = \langle v_m^2 v_n^2 \rangle - \langle v_m^2 \rangle \langle v_n^2 \rangle \]
- New and independent set of constraints on the system

Sensitivity to $\eta/s$ of QGP (not accessible with only one harmonic)

ALICE Collaboration, PRL 117, 182301 (2016)
Generalisation to three Different Harmonics

➢ Are there genuine correlations between more than two harmonics?
→ Possible new constraints for the system
(Not accessible with $\text{SC}(m, n)$ or only one harmonic)

“Higher order Symmetric Cumulants”

➢ Example: 3-harmonic Symmetric Cumulants:

$\text{SC}(k, l, m) = \langle v_k^2 v_l^2 v_m^2 \rangle - \langle v_k^2 v_l^2 \rangle \langle v_m^2 \rangle - \langle v_k^2 v_m^2 \rangle \langle v_l^2 \rangle - \langle v_l^2 v_m^2 \rangle \langle v_k^2 \rangle + 2 \langle v_k^2 \rangle \langle v_l^2 \rangle \langle v_m^2 \rangle$

$\text{SC}_\epsilon(k, l, m) = \langle \epsilon_k^2 \epsilon_l^2 \epsilon_m^2 \rangle - \langle \epsilon_k^2 \epsilon_l^2 \rangle \langle \epsilon_m^2 \rangle - \langle \epsilon_k^2 \epsilon_m^2 \rangle \langle \epsilon_l^2 \rangle - \langle \epsilon_l^2 \epsilon_m^2 \rangle \langle \epsilon_k^2 \rangle + 2 \langle \epsilon_k^2 \rangle \langle \epsilon_l^2 \rangle \langle \epsilon_m^2 \rangle$
Validation with Toy Monte Carlo Simulations

- Less than 3 correlated harmonics $\rightarrow$ $SC(2,3,4) = 0$

- 3 correlated harmonics $\rightarrow$ $SC(2,3,4) \neq 0$

![Graph showing SC(2,3,4) vs Multiplicity for less than 3 correlated harmonics and 3 correlated harmonics.](image1)

Toy Monte Carlo, $N_{\text{events}} = 1e7$

$v_2$, $v_3 = (0.05, 0.09)$, $v_4 = v_2 - 0.02$

![Graph showing SC(2,3,4) vs Multiplicity for 3 correlated harmonics.](image2)

Toy Monte Carlo, $N_{\text{events}} = 1e7$

$v_2 = (0.03, 0.1)$, $v_3 = v_2 - 0.02$, $v_4 = v_2 - (0.005, 0.025)$

Theoretical value: $5.13992e-9$
From Toy Simulations to Realistic Models

➢ Toy Monte Carlo simulations good verification tool…

… But what are the predictions for more realistic models?
From Toy Simulations to Realistic Models

➢ Toy Monte Carlo simulations good verification tool...

… But what are the predictions for more realistic models?

➢ Predictions from iEBE-VISHNU

- Event generator for heavy-ion collisions
- Describes the evolution of the collision
- Simulations → Access to $SC_{\epsilon}(k, l, m)$ and $SC(k, l, m)$
- 14000 simulated events per centrality bin → Feasibility test

Initial state from MC-Glauber model
After 0.6 fm/c, flow from the hydrodynamic evolution with $\eta/s = 0.08$
Predictions with iEBE-VISHNU

➢ System smaller for peripheral collisions → Harder to transfer the initial anisotropy to the final state
Predictions with iEBE-VISHNU

- System smaller for peripheral collisions $\rightarrow$ Harder to transfer the initial anisotropy to the final state

Cannot compare the initial and final states due to different scales
Normalised Symmetric Cumulants

- $\text{NSC}(k, l, m) = \frac{SC(k, l, m)}{\langle v_k^2 \rangle \langle v_l^2 \rangle \langle v_m^2 \rangle}$ and $\text{NSC}_\epsilon(k, l, m) = \frac{SC_\epsilon(k, l, m)}{\langle \epsilon_k^2 \rangle \langle \epsilon_l^2 \rangle \langle \epsilon_m^2 \rangle}$

- How much of the final particle correlations come from the initial state?

- Sign change between $\text{NSC}(2,3,4)$ and $\text{NSC}_\epsilon(2,3,4)$
  - SC(4,2) > 0
  - SC(3,2) < 0
Summary and Outlook

- Introduced the generalisation to higher order Symmetric Cumulants
- Validated SC($k,l,m$) with Toy Monte Carlo simulations
- Showed predictions from iEBE-VISHNU

- **Next step?** Analyses of SC($k,l,m$) on ALICE Pb-Pb data from RUN 1 and RUN 2
Backups
Parametrisations for $\eta/s$

Cumulants

- Cumulant \( \langle ... \rangle_c \) describes the genuine correlations between all the elements in the correlator
  - Term unique in the decomposition

- Cumulants not directly measurable → Must be computed from the correlations

- 2-particle cumulant: \( \langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle \)

- In 3-particle cumulant:
  \[
  \langle X_1 X_2 X_3 \rangle_c = \langle X_1 X_2 X_3 \rangle - \langle X_1 X_2 \rangle \langle X_3 \rangle - \langle X_1 X_3 \rangle \langle X_2 \rangle - \langle X_2 X_3 \rangle \langle X_1 \rangle + 2 \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle
  \]
  - Recursive method for higher order cumulants

- Study correlations
  - Different k-particle correlators: \( X_j = e^{in\phi_j} \) (N. Borghini, P. M. Dinh and J.-Y. Ollitrault, PRC 63, 054906 (2001))
  - Different harmonics: e.g. Symmetric Cumulants
Cumulants

- Cumulants not directly measurable ⇒ Must be computed from the correlations

- 2-particle cumulant: $\langle X_1X_2 \rangle_c = \langle X_1X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$

- In 3-particle cumulant:
  
  $\langle X_1X_2X_3 \rangle_c = \langle X_1X_2X_3 \rangle - \langle X_1X_2 \rangle \langle X_3 \rangle - \langle X_1X_3 \rangle \langle X_2 \rangle - \langle X_2X_3 \rangle \langle X_1 \rangle + 2 \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle$

- Recursive method for higher order cumulants

![Diagram of cumulants]
Requirements to be a Symmetric Cumulant

1) **No built-in trivial contribution**: if $v_k$ constant $\rightarrow SC(k,l,m) = 0$

2) **Genuine 3-harmonic correlations**: $SC(k,l,m) \neq 0$ only for 3 correlated harmonics

3) **Symmetry**: $SC(k,l,m) = SC(l,m,k) = SC(k,m,l) = \ldots$

4) **Cleanliness**: no dependence on symmetry planes in SC by definition

5) **Isotropy**: all correlators used in SC must be isotropic

6) **Uniqueness**: no ambiguity in the combination of correlators used to get the final expression

7) **Robustness against nonflow**: if only nonflow in the system $\rightarrow$ expected scaling of $SC(k,l,m)$

8) **Multiplicity weights**: which weights to use to go from $\langle \ldots \rangle$ to $\langle \langle \ldots \rangle \rangle$?

$\Rightarrow$ **How to check if $SC(k,l,m)$ follows the requirements?** *Toy Monte Carlo simulations*
Toy Monte Carlo Setup

- Controlled environment to test the influence of the initial parameters
- Setup implemented for this study:
  - Set the flow harmonics $v_n$ in the Fourier series
  - Uniform sampling of azimuthal angles $\varphi$ for $M$ particles
  - Compute the Q-vectors $Q_n$ for the event
  - Compute the needed correlators $\langle...\rangle$ from the Q-vectors for the event
  - Loop over $N$ events to compute $\langle\langle...\rangle\rangle$
  - Compute $SC(k,l,m)$

Initial: $v_n, M, N$

For one event: $\varphi, Q_n, \langle...\rangle$

Loop over events

For $N$ events: $\langle\langle...\rangle\rangle$

Final: $SC(k,l,m)$

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1) No built-in Trivial Contribution?

- $v_2, v_3$ and $v_4$ constant $\Rightarrow$ SC(2,3,4) = 0
- No built-in contribution
2) Genuine 3-harmonic Correlations?

- SC(2,3,4) compatible with 0 for all multiplicities

\[ SC(k,l,m) \] not sensitive to less than 3 correlated harmonics

Toy Monte Carlo, \( N_{\text{events}} = 1e7 \)
\( \nu_2, \nu_3, \nu_4 = (0.05, 0.09), \nu_4 = \nu_2 - 0.02 \)
2) Genuine 3-harmonic Correlations?

- Simple model of correlations among 3 harmonics: 
  \[ f(v_2, v_3, v_4) \text{ in: } \langle v_2^a v_3^b v_4^c \rangle = \int \int \int v_2^a v_3^b v_4^c f(v_2, v_3, v_4) dv_2 dv_3 dv_4 \]

- Compute all terms of SC(2,3,4)

- Simulations in agreement with theory for all multiplicities

- Sensitive to 3-harmonic correlations only

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6) Uniqueness in SC($m,n$)?

- Ambiguity in $\langle v_m^2 v_n^2 \rangle$ from $\langle \cos[n_1 \phi_1 + \cdots + n_k \phi_k] \rangle = v_{n_1} \cdots v_{n_k} \cos[n_1 \Psi_{n_1} + \cdots + n_k \Psi_{n_k}]$

- SC($m,n$) can be written as
  - Usual: $\langle \langle \cos(m(\phi_1 - \phi_2) + n(\phi_3 - \phi_4)) \rangle \rangle$
  - $- \langle \langle \cos(m(\phi_1 - \phi_2)) \rangle \rangle \langle \langle \cos(n(\phi_1 - \phi_2)) \rangle \rangle$
  - Alternative:
    $\langle \langle \cos(m(\phi_1 - \phi_2)) \rangle \rangle \langle \langle \cos(n(\phi_1 - \phi_2)) \rangle \rangle$
  - $- \langle \langle \cos(m(\phi_1 - \phi_2)) \rangle \rangle \langle \langle \cos(n(\phi_1 - \phi_2)) \rangle \rangle$

- From mathematical derivation:
  - $\langle \langle \cos(m(\phi_1 - \phi_2) + n(\phi_3 - \phi_4)) \rangle \rangle$ works
  - Autocorrelations in $\langle \langle \cos(m(\phi_1 - \phi_2)) \rangle \rangle \langle \langle \cos(n(\phi_1 - \phi_2)) \rangle \rangle$
6) Uniqueness?

➢ As for SC($m,n$), many expressions lead in theory to the same final expression for SC($k,l,m$)

   ☑ Again from $\langle \cos[n_1 \varphi_1 + \cdots + n_k \varphi_k] \rangle = v_{n_1} \cdots v_{n_k} \cos[n_1 \Psi_{n_1} + \cdots + n_k \Psi_{n_k}]$

➢ Example for the first term: $\langle v_k^2 v_l^2 v_m^2 \rangle$

   $\rightarrow \langle \cos(k(\varphi_1 - \varphi_2) + l(\varphi_3 - \varphi_4) + m(\varphi_5 - \varphi_6)) \rangle$?

   $\rightarrow \langle \cos(k(\varphi_1 - \varphi_2))\cos(l(\varphi_1 - \varphi_2))\cos(m(\varphi_1 - \varphi_2)) \rangle$?

   $\rightarrow \langle \cos(k(\varphi_1 - \varphi_2) + l(\varphi_3 - \varphi_4))\cos(m(\varphi_1 - \varphi_2)) \rangle$?

➢ Mathematical derivation too tedious for 3 different harmonics

   ☑ Use of the Toy Monte Carlo setup for 3 different expressions
6) Uniqueness?

- Usual: \(SC(k, l, m) = \langle 6 \rangle_{k,l,m,-k,-l,-m} - \langle 4 \rangle_{k,l,-k,-l} \langle 2 \rangle_{m,-m} - \langle 4 \rangle_{k,m,-k,-m} \langle 2 \rangle_{l,-l} - \langle 4 \rangle_{l,m,-l,-m} \langle 2 \rangle_{k,-k} + 2 \langle 2 \rangle_{k,-k} \langle 2 \rangle_{l,-l} \langle 2 \rangle_{m,-m}\)

- Alternative: \(SC(k, l, m) = \langle 2 \rangle_{k,-k} \langle 2 \rangle_{l,-l} \langle 2 \rangle_{m,-m} - \langle 2 \rangle_{k,-k} \langle 2 \rangle_{l,-l} \langle 2 \rangle_{m,-m} - \langle 2 \rangle_{k,-k} \langle 2 \rangle_{l,-l} \langle 2 \rangle_{m,-m} + 2 \langle 2 \rangle_{k,-k} \langle 2 \rangle_{l,-l} \langle 2 \rangle_{m,-m}\)

- Mixed: \(SC(k, l, m) = \langle 4 \rangle_{k,-l,-k,-l} \langle 2 \rangle_{m,-m} - \langle 2 \rangle_{k,-k} \langle 2 \rangle_{l,-l} \langle 2 \rangle_{m,-m} - \langle 2 \rangle_{k,-k} \langle 2 \rangle_{l,-l} \langle 2 \rangle_{m,-m} + 2 \langle 2 \rangle_{k,-k} \langle 2 \rangle_{l,-l} \langle 2 \rangle_{m,-m}\)
6) Uniqueness?

➢ Simulation with two anticorrelated harmonics
  ❖ Expected: zero for all multiplicities

➢ Usual: ok

➢ Alternative and mixed:
  ❖ Multiplicity dependence
  ❖ Presence of autocorrelations

➔ Unique way to write $\text{SC}(k,l,m)$ terms of azimuthal angles
7) Robustness against Nonflow?

- \[ SC(k, l, m) = \langle v_k^2 v_l^2 v_m^2 \rangle - \langle v_k^2 v_l^2 \rangle \langle v_m^2 \rangle - \langle v_k^2 v_m^2 \rangle \langle v_l^2 \rangle - \langle v_l^2 v_m^2 \rangle \langle v_k^2 \rangle + 2 \langle v_k^2 \rangle \langle v_l^2 \rangle \langle v_m^2 \rangle \]

- Expected behavior: \[ \delta_{SC(k, l, m)} \sim \frac{\alpha}{M^{6-1}} + \frac{\beta}{(M^{4-1})M^{2-1}} + \frac{\gamma}{(M^{2-1})^3} \]

- Simple nonflow with strong 2-particle correlations
  - Maximises nonflow
  - \[ \chi^2/\text{ndf} = 0.7 \]
  - Behavior as expected
    - \( \beta, \gamma \) compatible with zero
    - Dominant contribution to nonflow from 6-particle correlator

Nonflow scaling of a k-particle correlator: \[ \delta_k \sim \frac{1}{M^{k-1}} \]
7) Robustness against Nonflow?

Toy Monte Carlo, $N_{\text{events}} = 1 \times 10^8$
$v_2, v_3, v_4 = 0, M_{\text{final}} = 2 M_{\text{initial}}$

$\alpha/M^5 + \beta/M^4 + \gamma/M^3$
8) Multiplicity Weights?

➢ Which weights to pass from $\langle \ldots \rangle$ to $\langle \langle \ldots \rangle \rangle$?

- Number of combinations for a k-particle correlator: $\prod_{i=0}^{k-1} (M - i)$
- Multiplicity of the event $M$
- Unit weight

➢ $M$ uniformly sampled in (50, 500)

➢ Constant harmonics

➔ Smallest statistical spread for number of combinations

- Same conclusion as for $SC(m,n)$
Realistic Monte Carlo studies

- Used **two Monte Carlo models**: HIJING and iEBE-VISHNU

- Heavy-Ion Jet INteraction Generator (HIJING):
  - Model to study particle and jet production in nuclear and heavy-ion collisions
  - Description of phenomena with correlations between few particles ➔ Only nonflow
  - Used to test the robustness of SC\((k,l,m)\) against nonflow

- iEBE-VISHNU:
  - Heavy-ion collision event generator based on hydrodynamics calculations
  - Describes heavy-ion collision evolution
  - Contains both flow and nonflow, only flow used in this work

- For both, simulations of Pb-Pb collisions with \(\sqrt{s} = 2.76\) TeV
HIJING Simulations

➢ SC($k,l,m$) compatible with zero for head-on and mid-central collisions ➔ Not sensitive to nonflow
Cross-Check of VISHNU with ALICE Data

- Comparison of ALICE 2010 Pb-Pb data with VISHNU simulations
- Sensitivity of SC on the lower limit of the $p_T$ range
  - $0.28 < p_T < 4$ GeV: better qualitative agreement with data
  - ALICE $p_T$ range: $0.2 < p_T < 5$ GeV
- Dependence of SC(3,2) on $\eta/s$ not studied here
  - Already studied in ALICE Collaboration, PRL 117, 182301 (2016)
Comparison between HIJING and VISHNU

- Values from HIJING smaller than the ones from VISHNU
- Results from VISHNU not a systematic bias of HIJING

Any nonzero results of SC in real data ➔ Collective flow effects